

# Vacuum polarization in the Szekeres Class of colliding plane wave space-times

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## Abstract

We study the quantization of a scalar field on a classical background given by the *Szekeres Class of solutions*, which represent the collision of two gravitational plane waves with constant polarization. These solutions consist of two approaching gravitational plane waves moving in a flat background and an interaction region which always contains a curvature singularity. Following a suitable approximate procedure, introduced in a previous paper, we propose a way to compute the vacuum expectation value of the stress-energy tensor throughout the causal past region of the collision center in the quantum state which corresponds to the vacuum before the arrival of the waves.

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# 1 introduction

Gravitational waves in General Relativity are considered as perturbations of the space-time geometry which propagate with the speed of light. Contrarily to the behaviour of electromagnetic waves in flat space, when gravitational waves pass through each other there is a non-linear interaction. Of course, such a difference arise from the fact that, unlike Maxwell's equations, Einstein's equations are highly non-linear. The situation is very different in the linearized theory of gravity where gravitational waves superpose without interaction. Even in the full Einstein-Maxwell theory, Maxwell's equations remain linear indicating non direct electromagnetic interaction between two pure electromagnetic waves. In that case, however, there is a non-linear interaction of the waves through the gravitational field generated by their electromagnetic energy. Although very small, the magnitude of such an interaction is similar to the magnitude of the interaction between pure gravitational waves.

In order to find exact solutions of Einstein's equations describing plane wave collisions it is necessary to introduce several simplifying assumptions. Perhaps, the most restrictive simplification would be the assumption that the interacting waves have plane symmetry. Indeed, exact *gravitational plane waves* are very simple time dependent plane symmetric solutions of Einstein's equations [1]. That may not seem a very restrictive assumption if we intuitively think of plane fronted waves as approximations to spherical waves at large distances from their sources. However, the global features of the two type of waves are very different. In fact, exact gravitational plane waves exhibit two main surprising global features, namely: i) the absence of a global Cauchy surface, which is a consequence of the focusing effect that the waves exert on null rays [2], ii) the presence of a coordinate singularity on some hypersurface behind the wave. Such a singularity may be physically understood in terms of *caustics* that are formed in the regions where the null rays are focused [3]. For pure gravitational plane waves, a whole three-dimensional congruence of parallel null geodesics may be focused into a line. For pure electromagnetic plane waves, the region of focusing may be a simple event. From this property it follows that we can conveniently consider the inverse of the focusing time as a measure of the strength of the wave. In particular, for a pure electromagnetic wave such inverse time equals the electromagnetic energy per unit surface of the wave.

Beyond the assumption that the wave front is plane, the imposition of plane symmetry also requires that the magnitude of the wave is constant all over the entire plane. Furthermore, if we only consider *head on* collisions, then it makes sense to impose the condition of plane symmetry globally. Actually, it may seem that there is not loss of generality if we concentrate only on head on collisions. The reasons are that the particular case of plane waves propagating in the same direction is trivial since the waves do not interact, they simply superpose linearly [4, 5]. Also, we may think that it is always possible to make a Lorentz transformation in order to include oblique collisions. However, the assumption of global plane symmetry would severely blur the physical interpretation of the solutions obtained in this particular way.

Nevertheless, besides their simplicity, one expects that exact plane waves may be relevant for the study of the strong time dependent gravitational fields that may be produced in the collision of black holes [6, 7] or to represent travelling waves on strongly gravitating cosmic strings [8]. In recent years these waves have been used in classical general relativity to test some conjectures on the stability of Cauchy horizons [9, 10], and in string theory to test

classical and quantum string behaviour in strong gravitational fields [11, 12, 13]. Their interest also stems from the fact that plane waves are a subclass of exact classical solutions to string theory [14, 15, 16].

It is known that when waves are coupled to quantum fields there is neither vacuum polarization nor the spontaneous creation of particles. In that sense they behave very much as electromagnetic or Yang-Mills plane waves in flat space-time [17, 18]. Still the classical focusing of geodesics has a quantum counterpart: when quantum particles are present the quantum field stress-energy tensor between scattering states is unbounded at the Cauchy horizon, i.e. where classical test particles focus after colliding with the plane wave [19]. This suggests that the Cauchy horizon of plane waves may be unstable under the presence of quantum particles. The classical instability of the null Cauchy horizons of plane waves is manifest when non-linear plane symmetric gravitational radiation collides with the background wave, i.e. when *two plane waves collide*. In this case the focusing effect of each wave distorts the causal structure of the space-time near the previous null horizons and either a spacelike curvature singularity or a new regular Killing-Cauchy horizon is formed. However, it is generally believed that the Killing-Cauchy horizons of the colliding plane wave space-times are unstable in the sense that “generic” perturbations will transform them into spacelike curvature singularities. In fact, this has been proved under general plane symmetric perturbations [20]. Also exact colliding plane wave solutions with classical fields are known that have spacelike curvature singularities and which reduce, in the vacuum limit, to colliding plane wave solutions with a regular Killing-Cauchy horizon [21].

Note that the singularities derived from plane wave collisions are not the result of the collapse of matter but the result of the non-linear effects of pure gravity. Note also that the presence of a Killing-Cauchy horizon in a colliding plane wave space-time implies a breakdown of predictability since the geometry beyond the horizon is not uniquely determined by the initial data posed by the incoming colliding waves. In fact, these extensions exhibit a wide variety, they may or may not contain space-like curvature singularities. Even they may contain time-like singularities, which can be avoided for certain observers in a space-like motion (see [22] for a review on the subject).

The interaction of quantum fields with colliding plane waves was first considered by Yurtsever [23] for the singular Khan-Penrose solution [24], which describes the collision of two plane impulsive gravitational waves. In that case, an unambiguous “out” vacuum state was possible to define in a relatively simple way. More recently, Dorca and Verdaguer [25, 26] noticed that the presence of a Killing-Cauchy horizon in a non singular colliding plane wave space-time could be used to define an unambiguous “out” vacuum state related to the preferred Hadamard state introduced by Kay and Wald in more generic space-times with Killing-Cauchy horizons [27]. With this premise, Dorca and Verdaguer studied the interaction of quantum fields in a particular non-singular colliding plane wave space-time, the interaction region of which was isometric to a region inside the event horizon of a Schwarzschild black hole [28, 29]. Later on the same premise was applied by Feinstein and Sebastián [30] to the Bel-Szekeres solution [31], which represents the head on collision of two electromagnetic plane waves with an interaction region isometric to the Bertotti-Robinson universe [32] filled with an uniform electric field. In all these examples it was found that the initial state, defined to be the vacuum state in the flat region before the arrival of the waves, contained a spectrum of “out” particles consistent, in the long wavelength limit, with

a thermal spectrum with a temperature inversely proportional to the focusing time of the waves.

A further step in the study of the interaction of quantum fields with colliding plane waves is the computation of the expectation value of the stress-energy tensor. However, the non perturbative evaluation of the stress-energy tensor of a quantum field in a dynamically evolving space-time is generally a difficult task. Even when the exact modes of the quantum field equation are known it may not be possible to perform the mode sums in order to get the quantum field two point function or, more precisely, the Hadamard function, which is the key ingredient in the evaluation of the stress-energy tensor. This problem was first considered by Yurtsever [23] for the Khan-Penrose solution [24]. In that case it was possible to determine the behaviour of the stress-energy tensor near the singularity of the interaction region. It was shown that for the conformal coupling case (i.e.  $\xi = 1/6$ ) the energy density and two of the principal pressures were positive and unbounded towards the singularity. This problem has been also considered by Dorca and Verdaguer in the mentioned above non-singular colliding plane wave spacetime with an interaction region isometric to an interior region of a Schwarzschild black hole. As in the case of Yurtsever for the Khan-Penrose solution, the expectation value of the stress-energy tensor was calculated in the state representing the Minkowski vacuum in the flat region before the arrival of the waves. This value was first computed in a region close to both the Killing-Cauchy horizon and the topological singularities, the *folding singularities*, that the colliding plane wave space-time contains [22]. In that particular region, the calculations were simplified due to the blueshift effect on the energy of the initial quantum modes as they reached the Killing-Cauchy horizon [33]. In a recent work [34], an approximation procedure was proposed by the author in order to calculate such an expectation value throughout the causal past region of the collision center. That approximation has been also applied by the author in the Bel-Szekeres space-time [35]. In all of these calculations, it was found that the stress-energy diverged as the Killing-Cauchy horizon was approached. The rest energy density was positive and unbounded towards the horizon. Two of the principal pressures were negative and of the same order of magnitude of the energy density. It was also pointed out that such a behaviour suggested that the non singular Killing-Cauchy horizon is indeed unstable under quantum perturbations and a curvature singularity would be the general outcome of a generic plane wave space-time when backreaction is taking into account. Note that this is a non perturbative effect, it is the result of the nonlinearity of gravity, since gravitational waves in the linear approximation do not polarize the vacuum. In fact the vacuum stress-energy tensor of a quantum field in a weakly inhomogeneous background was computed by Horowitz [36], and it is easy to see that such tensor can be written in terms of the linearized Einstein tensor only [37], which vanishes for gravitational waves [38, 39].

The plan of the paper is the following. In section 2 the geometry of plane fronted waves and colliding plane waves is briefly reviewed. In section 3 the mode solutions of the scalar field equation are given throughout the causal past of the collision center in the four different regions of the space-time. It is explicitly seen that an exact expression for these mode solutions can be easily found everywhere but in the interaction region. However, following the directions of two previous works [34, 35], we will give an adequate approximation for the field solutions in the interaction region. Such an approximation will allow us to recover the singularity pattern of a Hadamard function, which is essential in order to perform a

correct regularization. In section 4 a brief summary of the main *point-splitting* regularization formulae is given. Finally, in section 5, a procedure to calculate the Hadamard function is introduced. Also, are given the basic details to compute the vacuum expectation value of the stress-energy tensor in the causal region of the collision center. A summary and some consequences of the results are discussed in section 6. Also, some bitensor covariant expansions have been stored in the Appendices.

## 2 Description of the geometry

In this section we start with a review of the geometrical properties of gravitational plane waves, and we follow with a geometrical analysis of the head on collision of two gravitational plane waves.

### 2.1 Gravitational and electromagnetic plane waves

The appropriate notation to describe plane waves is that of Newman and Penrose (1962) [40]. We introduce a tetrad on null vectors, namely, two real vectors  $n^\mu$ ,  $l^\mu$ , and a complex null vector  $m^\mu$  and its complex conjugate  $\bar{m}^\mu$ , such that  $n^\mu l_\mu = 1$ ,  $m^\mu \bar{m}_\mu = -1$ , and they satisfy the completeness relation,

$$g_{\mu\nu} = l_\mu n_\nu + n_\mu l_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu.$$

Then we can write the ten independent components of the Ricci and the ten independent components of the Weyl tensor in this tetrad basis. This turns to be physically convenient since the components of the Weyl tensor in the tetrad basis have a well established meaning of Coulomb components or transverse and longitudinal wave components in the null directions  $n^\mu$ ,  $l^\mu$  [41]. This interpretation is also very useful to analyze the interaction of two of such waves since it is possible to align the basis vectors  $n^\mu$  and  $l^\mu$  with the propagation directions of the two waves. Furthermore, if we consider the interaction between transverse waves, the problem reduces to find the interaction between the respective transverse Weyl components.

In the context of plane waves we introduce the class of *pp-waves*, i.e plane fronted gravitational waves with parallel rays. They are defined by the property that they admit a covariantly constant null vector field, which is possible to interpret as the rays of the wave and it may be identified with the tetrad vector  $l^\mu$ . There exist a family of 2-surfaces orthogonal to  $l^\mu$  that may be interpreted as wave surfaces [42]. It is convenient to use a null coordinate  $u$ , defined by  $l_\mu = u_{,\mu}$ , and the metric can be written in the *Kerr-Shild form* as,

$$ds^2 = 2 du dr + H(u, X, Y) du^2 - dX^2 - dY^2, \quad (1)$$

where the coordinates  $X$  and  $Y$  span the wave surfaces. These type of metrics are either of algebraic type N, or conformally flat and the only non-zero component of the Weyl tensor, in the tetrad basis, is the transverse wave component in the  $l^\mu$  direction. The modulus term of this component can be conveniently interpreted as the *amplitude* of the wave and its phase term as the *polarization* of the wave.

If we suppose that these space-times are solutions of Einstein's equations, the function  $H$  in the metric (1) must satisfy the differential equation,

$$H_{,XX} + H_{,YY} = 0.$$

Recall that this is a linear differential equation which means that distinct solutions of pp-waves (1) may be superposed, i.e. two independent pp-waves, propagating in the same direction do not interact [4, 5]. Another remarkable property of pp-waves, which was pointed out by Yurtsever [43], states that any gravitational wave space-time that is flat before the arrival of the wave and returns to a perfect flatness after the wave passes, i.e. a *sandwich wave*, it is necessarily a pp-wave.

In *Einstein-Maxwell theory*, the particular class of plane-symmetric waves are defined to be pp-waves in which the curvature field components are the same at every point of the wave surfaces, i.e. the transversal  $X, Y$ -planes. There is only one non-null component of the Ricci tensor and, as mentioned above, only one non-null component of the Weyl tensor. Both components depend only on second derivatives of the function  $H(u, X, Y)$  with respect to the transversal coordinates  $X$  and  $Y$ . Therefore, the plane symmetry condition implies that  $H(u, X, Y)$  is quadratic in  $X$  and  $Y$ , i.e.,

$$H(u, X, Y) = h_{11}(u)X^2 + 2h_{12}(u)XY + h_{22}(u)Y^2.$$

In that case the non null components of the Ricci tensor, in the tetrad basis, is given by the combination  $(h_{11} + h_{22})/2$  and the non null component of the Weyl tensor, in the tetrad basis, i.e. the transversal wave component in the direction  $l^\mu$ , is given by the combination,  $\frac{1}{2}(h_{11} - h_{22} + i2h_{12})$ . Then, depending on the values of  $h_{ij}(u)$ , we will distinguish between two type of waves, namely, i) *pure gravitational waves*, when  $h_{11} = -h_{22}$ , and therefore the Ricci tensor is exactly zero and we have a vacuum solution of Einstein's equations. If in addition  $h_{11}$  is proportional to  $h_{12}$ , then the wave has constant linear polarization. ii) *pure electromagnetic waves* when  $h_{11} = h_{22}$  and  $h_{12} = 0$ , and therefore the Weyl tensor is exactly zero.

In order to consider the collision and interaction of plane waves it is first convenient to change to *Rosen-like* coordinates  $(u, v, x, y)$ , where  $u$  and  $v$  are two null coordinates,  $x$  and  $y$  two transverse coordinates, and the line element (1) can be transformed into,

$$ds^2 = 2dudv - e^{-U} \left( e^V \cosh W dx^2 - 2 \sinh W dx dy + e^{-V} \cosh W dy^2 \right), \quad (2)$$

where  $U$ ,  $V$  and  $W$  are functions of  $u$  only. In the case of linear polarization, it is always possible to set  $W = 0$ .

However, the line element (2) in Rosen form, always contains a coordinate singularity on some hypersurface behind the wave. This singularity appear as a consequence of the focusing effect which plane waves exert on null geodesics. Physically it may be interpreted as a *caustic* produced in the region of null ray focusing [3]. Related to this focusing property of plane waves, as was first pointed out by Penrose [2], there is the fact that plane wave space-times do not contain global Cauchy surfaces, i.e it is not possible to set up initial values for a plane wave on any global space-like surface which lies entirely at the causal past of the wave front.

## 2.2 Colliding waves

The colliding plane wave space-time consists of two approaching waves, regions II and III, in a flat background, region IV, and an interaction region, region I. The two waves move in the direction of two null coordinates  $u$  and  $v$ , and since they have plane symmetry in the direction of the transversal coordinates  $x$  and  $y$ , the interaction region retains a two-parameter symmetry group of motions generated by the Killing vectors  $\partial_x$  and  $\partial_y$ . At each point of the interaction region, the null directions, along coordinates  $u$  and  $v$ , are orthogonal to the planes spanned by  $\partial_x$  and  $\partial_y$ . We may align the tetrad vectors  $n^\mu$  and  $l^\mu$  with these two null directions in such a way that,  $l_\mu = u_{,\mu}/L$  and  $n_\mu = v_{,\mu}/N$ , being  $L$  and  $N$  two functions which do not depend on coordinates  $x$  and  $y$ .

The general line element for the interaction region can be written in *Szekeres form* as

$$ds^2 = 2e^{-M}dudv - e^{-U} \left( e^V \cosh W + 2 \sinh W dx dy + e^{-V} \cosh W dy^2 \right), \quad (3)$$

where  $M$ ,  $U$ ,  $V$  and  $W$  are functions of  $u$  and  $v$  only. These functions are constrained by Einstein's equations together with a suitable set of initial conditions at the boundaries with the plane wave regions II and III. It is possible to set in general,

$$e^{-U} = a(u) + b(v), \quad (4)$$

with  $a(u)$  and  $b(v)$ , two arbitrary *monotonically decreasing* functions that may be considered piecewise  $C^1$ . Recall that the line element (3) is rather convenient because it is similar to that for a single plane wave in Rosen form (2), except that the functions  $M$ ,  $U$ ,  $V$  and  $W$  are functions of both coordinates  $u$  and  $v$  in region I, of coordinate  $u$  alone in region II and of coordinate  $v$  alone in region III. In region IV all these functions are constant and may be set to be zero by an appropriate coordinate parametrization.

The appropriate junction conditions across the null hypersurfaces  $u = 0$  and  $v = 0$ , are those of O'Brian and Synge [44]. These conditions demand that only the components,  $g_{\mu\nu}$ ,  $g^{ij}g_{ij,0}$ ,  $g^{i0}g_{ij,0}$ , should be continuous through the null hypersurfaces. Notice that O'Brian and Synge conditions are weaker than the usual Lichnerowicz conditions, which require that the metric tensor is  $C^1$  and piecewise at least  $C^2$ . In particular, Lichnerowicz conditions exclude the possibility of *impulsive gravitational waves*.

The function  $U$  is appropriately set by the boundary conditions (see for instance [22]):

$$a(u) = \frac{1}{2}; \quad u \leq 0, \quad \text{and} \quad \dot{a}(0) = 0,$$

$$b(v) = \frac{1}{2}; \quad v \leq 0, \quad \text{and} \quad \dot{b}(0) = 0.$$

Thus, the values of  $\exp(-U)$  are 1 in region IV,  $1/2 + b(v)$  in region III,  $1/2 + a(u)$  in region II and  $a(u) + b(v)$  in region I. The statement that  $U$  is an smooth function across the null boundaries requires that the function  $a(u)$  must have the same form in both regions I and II. Similarly, the function  $b(v)$  must have the same form in regions I and III. Furthermore, being  $a(u)$  and  $b(v)$  monotonically decreasing functions, it is always possible to express them in the form,

$$a(u) = \frac{1}{2} - \Theta(u) (c_1 u)^{n_1}, \quad b(v) = \frac{1}{2} - \Theta(v) (c_2 v)^{n_2}, \quad (5)$$

for a certain positive values of the parameters  $c_1, c_2, n_1, n_2$  and where  $\Theta(\zeta)$  is the usual step function. Recall that this expressions for  $a(u)$  and  $b(v)$  apply globally. Observe that, since functions  $a(u)$  and  $b(v)$  are monotonically decreasing, it is inevitable that a singularity, either a curvature singularity or a Killing Cauchy horizon, at  $a(u) + b(v) = 0$  is formed. The two constants  $c_1$  and  $c_2$  in (5) are a measure of the strength of the waves. In fact, the larger these parameters are, the sooner the singularity is produced. It is also possible to use further rescaling on null coordinates  $u$  and  $v$  to set  $c_1 = c_2 = 1$ , and with this rescaling, as depicted in Fig. 1, we have a symmetric wave collision. Once a rescaling is used, to set  $c_1 = c_2 = 1$ , then a measure of the strength of the waves is absorbed into the function  $\exp(-M)$  in line element (3). In fact,  $\exp(-M)$  will be proportional to  $L_1 L_2$ , where the two new parameters  $L_1$  and  $L_2$  are directly related to  $c_1$  and  $c_2$  as  $L_1 = c_1^{-1}$  and  $L_2 = c_2^{-1}$ .

Once a value for function  $U$  is set, Einstein's equations together with suitable initial conditions at the boundaries with the null hypersurfaces  $u = 0$  and  $v = 0$ , will determine the values for the remaining functions, i.e  $V, M$  and  $W$ . For the particular case of linearly polarized plane waves, we may set  $W = 0$ . In that case, one possible family of solution of Einstein's equations which satisfy the boundary conditions is the *Szekeres Class of solutions* [45], for which,

$$e^{V(u,v)} = (a+b)^{p_1+p_2} \left( \sqrt{1/2+b} + \sqrt{1/2-a} \right)^{-2p_1} \left( \sqrt{1/2+a} + \sqrt{1/2-b} \right)^{-2p_2}, \quad (6)$$

$$e^{-M(u,v)} = \frac{(a+b)^{[(p_1+p_2)^2-1]/2}}{(1/2+a)^{p_2^2/2}(1/2+b)^{p_1^2/2}} \left( \sqrt{1/2-a}\sqrt{1/2-b} + \sqrt{1/2+a}\sqrt{1/2+b} \right)^{-2p_1p_2}, \quad (7)$$

where  $p_1$  and  $p_2$  are two real parameters related to the parameters  $n_1$  and  $n_2$  in the definition of functions  $a(u)$  and  $b(v)$  (5) by,

$$p_1^2 = 2 \left( 1 - \frac{1}{n_1} \right), \quad p_2^2 = 2 \left( 1 - \frac{1}{n_2} \right). \quad (8)$$

In order to satisfy the boundary conditions, it is necessary that,  $n_i \geq 2$  with  $i = 1, 2$ , and therefore the constants  $p_i$  are restricted to the range of values  $1 \leq p_i^2 < 2$ . For instance, the particular values  $p_1 = p_2 = 1$  correspond to the Khan-Penrose solution [24]. In this family of solutions, there is always a scalar polynomial curvature singularity in the interaction region on the surface  $a(u) + b(v) = 0$ .

We will restrict in the following sections to the Szekeres Class of solutions. The colliding plane wave space-times can be briefly described as follows (see Fig. 1): The general space-time contains four regions, namely: two single plane wave regions (regions II and III) moving in a Minkowski flat background (region IV) and an interaction region (region I). These four space-time regions are separated by the two null wave fronts  $u = 0$  and  $v = 0$ . Namely, the boundary between regions I and II is  $\{0 \leq u < 1, v = 0\}$ , the boundary between regions I and III is  $\{u = 0, 0 \leq v < 1\}$ , and the boundary between regions II and III with region IV is  $\{u \leq 0, v = 0\} \cup \{u = 0, v \leq 0\}$ . The interaction region is bounded by a scalar polynomial curvature singularity at the surface  $a(u) + b(v) = 0$ . Region I meets region IV only at the surface  $u = v = 0$ , and the plane wave region II or III meets the



singularity of region I at  $\mathcal{P} = \{u = 1, v = 0\}$  or  $\mathcal{P}' = \{u = 0, v = 1\}$ , respectively. Observe that, the single plane wave regions II and III always contain a singularity at the hypersurface  $u = 1$  for region II and  $v = 1$  for region III. For the particular case of the Khan and Penrose solution, which corresponds to the values  $n_1 = n_2 = 2$ , these singularities are just coordinate singularities. For all the other cases, i.e.  $n_1, n_2 > 2$ , these singularities are non-scalar curvature singularities [46]. However, in what follows we will refer generically to them as *folding singularities*. The reason for this terminology comes from the behaviour of null geodesics near them. Namely, only a null measure set of null geodesics which enter into the plane wave regions II or III hit the singularities, all the rest are *folded* to end up into the interaction region. Such a behaviour suggest that the whole singularity at  $u = 1$  in region II (or  $v = 1$  in region III) should be identified with the surface  $\mathcal{P} = \{u = 1, v = 0\}$  (or  $\mathcal{P}' = \{u = 0, v = 1\}$ ) in region I (see for instance [25] for a 3-dimensional plot of a space-time of this type).

### 3 Mode propagation

For simplicity we will consider in this section a massless scalar field, which satisfies the usual Klein-Gordon equation,

$$\square\phi = 0. \tag{9}$$

Following the directions of the approximation procedure introduced in the previous works [34, 35], we will be interested in the value of the quantum field  $\phi$  all over the causal past region of the collision center. The reason is essentially because the calculations can be greatly simplified in this region. We will start with the field solution in the flat region prior to the arrival of the waves, which is chosen to be the usual vacuum in Minkowski space-time. This vacuum solution will set a well posed initial value problem on the null boundary  $\Sigma = \{u = 0, v \leq 0\} \cup \{u \leq 0, v = 0\}$ , by means of which a unique solution for the field equation can be found throughout the space-time, i.e., in the plane wave regions (regions II and III), and in the interaction region (region I). However, although it is rather easy to find the solution of the field equation in regions II and III which matches smoothly with the boundary conditions, it turns out to be a difficult problem for the interaction region. The reason is essentially due to the intrinsic differences between the geometry of the plane wave regions and the interaction region. In fact, as mentioned in section 2, the plane wave regions are either conformally flat or type N in the Petrov classification, but the interaction region can be more generic. We will refer, from now on, to this problem as *the mode propagation problem*.

We will consider the line element,

$$ds^2 = 2e^{-M(u,v)}dudv - e^{-U(u,v)}\left(e^{V(u,v)}dx^2 + e^{-V(u,v)}dy^2\right), \tag{10}$$

which applies globally to the four space-time regions, and where the functions  $U$ ,  $V$  and  $M$ , can be directly read off (4)-(7). Then, the field equation can be separated in a plane-wave form solution for the transversal coordinates  $x$  and  $y$ , with  $k_x$  and  $k_y$ , respectively, as separation constants. This plane-wave separation is just a trivial consequence of the

translational symmetry of the space-time on the planes spanned by the Killing vectors  $\partial_x$  and  $\partial_y$ . The field solution is thus,

$$\phi(u, v, x, y) = e^{U(u,v)/2} f(u, v) e^{ik_x x + ik_y y}, \quad (11)$$

where the function  $f(u, v)$  satisfies the following second order differential equation,

$$f_{,uv} + \Omega(u, v) f = 0; \quad \Omega(u, v) = -\frac{(e^{-U/2})_{,uv}}{e^{-U/2}} + \frac{1}{2} e^{-M+U} (k_x^2 e^{-V} + k_y^2 e^V). \quad (12)$$

From now on, we will refer to  $\Omega(u, v)$  as the *potential term*. In the flat region (region IV) this potential term is simply,

$$\Omega_{\text{IV}}(u, v) = L_1 L_2 (k_x^2 + k_y^2), \quad (13)$$

where we have used that the functions  $U$  and  $V$  in (10) are zero in the flat region, and the function  $\exp(-M) = 2L_1 L_2$ . Using (13), equation (12) can be solved as,

$$f(u, v) = e^{-i2\hat{k}_+ u - i2\hat{k}_- v}, \quad (14)$$

where  $k_{\pm}$  are two new separation constants with dimensions of energy, and we define for convenience two dimensionless constants as  $\hat{k}_{\pm} \equiv \sqrt{L_1 L_2} k_{\pm}$ . These new separation constants are directly related to the previous ones  $k_x$  and  $k_y$  by,

$$4k_+ k_- = k_x^2 + k_y^2. \quad (15)$$

The field solution in region IV reduces, thus, to the usual Minkowski plane wave solution, i.e.,

$$\phi_k(u, v, x, y) = \frac{1}{\sqrt{2k_-}(2\pi)^3} e^{-i2\hat{k}_+ u - i2\hat{k}_- v + ik_x x + ik_y y}. \quad (16)$$

These modes are well normalized on the null hypersurface  $\Sigma$ , which is the boundary of the plane wave regions II and III with the flat region IV. Even though  $\Sigma$  is a null hypersurface, a well defined scalar product is given by (see [25] for details),

$$(\phi_1, \phi_2) = -i \int dx dy \left[ \int_{-\infty}^0 \left( \phi_1 \overleftrightarrow{\partial}_u \phi_2^* \right) \Big|_{v=0} du + \int_{-\infty}^0 \left( \phi_1 \overleftrightarrow{\partial}_v \phi_2^* \right) \Big|_{u=0} dv \right]. \quad (17)$$

The modes (16) will determine on  $\Sigma$  a well posed set of boundary conditions for modes in regions II and III. There, the potential term in equation (12) is simply,

$$\Omega_i(u, v) = \frac{1}{2} e^{-M_i+U_i} (k_x^2 e^{-V_i} + k_y^2 e^{V_i}), \quad (18)$$

where the label  $i = \text{II}$  or  $i = \text{III}$  in the functions  $U$ ,  $V$  and  $M$ , stands for their particular values in the plane wave regions II or III. Then, the solution of equation (12) in regions II and III with the boundary conditions imposed by the flat modes (16) on the hypersurface  $\Sigma$ , can be easily found as,

$$f(u, v) = \begin{cases} e^{-i2\hat{k}_- v - iA_{\text{II}}(u)/(2\hat{k}_-)}; & \text{in region II,} \\ e^{-i2\hat{k}_+ u - iA_{\text{III}}(v)/(2\hat{k}_+)}; & \text{in region III,} \end{cases} \quad (19)$$

where the generic function  $A_i(\zeta)$ , with  $i = \text{II, III}$  is given by,

$$A_i(\zeta) = \int_0^\zeta d\zeta' \frac{1}{2} e^{-M_i(\zeta') + U_i(\zeta')} \left( e^{-V_i(\zeta')} k_x^2 + e^{V_i(\zeta')} k_y^2 \right). \quad (20)$$

Therefore, the well normalized “in” modes in regions II and III are,

$$\phi(u, v, x, y) = \frac{1}{\sqrt{2k_-}(2\pi)^3} e^{ik_x x + ik_y y} \begin{cases} \frac{1}{\cos u} e^{-i2\hat{k}_- v - iA_{\text{II}}(u)/(2\hat{k}_-)}; & \text{in region II,} \\ \frac{1}{\cos v} e^{-i2\hat{k}_+ u - iA_{\text{III}}(v)/(2\hat{k}_+)}; & \text{in region III,} \end{cases} \quad (21)$$

Now the Cauchy problem is well posed on the boundary  $\Sigma_{\text{I}} = \{u = 0, 0 \leq v < 1\} \cup \{0 \leq u < 1, v = 0\}$  between plane wave regions II and III and interaction region. Notice that the initial modes (21) are well normalized on the boundary  $\Sigma$  between the flat region and the plane wave regions, and this means, from general grounds, that they remain well normalized on the boundary between the plane waves and the interaction region. This can be seen explicitly using a well defined scalar product on the hypersurface  $\Sigma_{\text{I}}$ , which similarly to (17) is,

$$(\phi_1, \phi_2) = -i \int dx dy \left[ \int_0^1 e^{-U_{\text{II}}(u)} \left( \phi_1 \overleftrightarrow{\partial}_u \phi_2^* \right) \Big|_{v=0} du + \int_0^1 e^{-U_{\text{III}}(v)} \left( \phi_1 \overleftrightarrow{\partial}_v \phi_2^* \right) \Big|_{u=0} dv \right]. \quad (22)$$

The correct normalization of modes (21) follows easily from the fact that functions  $A_i(x)$  in (20) are unbounded at the folding singularities,  $\mathcal{P} = \{u = 1, v = 0\}$  and  $\mathcal{P}' = \{u = 0, v = 1\}$ .

We have now to solve equation (12) in region I with the boundary conditions imposed by (21) on the lines  $\Sigma_{\text{I}} = \{u = 0, 0 \leq v < 1\} \cup \{0 \leq u < 1, v = 0\}$ , which are characteristic lines for the partial differential equation (12). Thus, the only independent boundary conditions, i.e. the Cauchy data, are the initial values of the function  $f(u, v)$  on them (the normal derivatives of the field on the characteristics, which are usually part of the Cauchy data, are determined by the values of the function  $f(u, v)$  itself, see [47] for details). Furthermore, if we recall that the collision center is determined by the condition  $u = v$ , then the subset of Cauchy data that affects the neighborhood of the collision center, i.e its causal past, lies on the segments,  $\hat{\Sigma}_{\text{I}} = \{u = 0, 0 \leq v < \hat{s}\} \cup \{0 \leq u < \hat{s}, v = 0\}$  [47], where  $\hat{s}^{n_1} + \hat{s}^{n_2} = 1$  and  $n_1, n_2$  are defined in (8). We will denote the causal future region of these Cauchy data (or equivalently, the causal past region of the collision center) by region  $\mathcal{S}$ , see Fig. 2.

We can determine the behaviour of the solutions of equation (12) in the region near the singularity using the following change of coordinates,

$$\sigma = b(v) + a(u), \quad \rho = b(v) - a(u). \quad (23)$$

Then equation (12) becomes,

$$f_{,\sigma\sigma} - f_{,\rho\rho} + \Omega(\sigma, \rho)f = 0, \quad (24)$$

where the potential term is,

$$\Omega(\sigma, \rho) = \frac{1}{4\sigma^2} \left[ 1 + 2e^{-\hat{M}} \frac{\sigma^{\alpha-\beta+1}}{\dot{a}\dot{b}} \left( e^{-\hat{V}} k_1^2 + \sigma^{2\beta} e^{\hat{V}} k_2^2 \right) \right], \quad (25)$$

and where we have used for convenience two dimensionless separation constants  $k_1$  and  $k_2$ , directly related to  $k_x$  and  $k_y$  by,

$$k_1 = \sqrt{L_1 L_2} k_x, \quad k_2 = \sqrt{L_1 L_2} k_y. \quad (26)$$

Observe that this equation has a singular point at  $\sigma = 0$  due to the term  $\sigma^{-2}$  on the potential  $\Omega(\sigma, \rho)$ . In fact, we have written the functions  $\exp(-M)$  and  $\exp(\pm V)$  in the following way,

$$e^{-M} = \sigma^\alpha e^{-\hat{M}}, \quad e^V = \sigma^\beta e^{\hat{V}}, \quad \alpha = (\beta^2 - 1)/2, \quad \beta = p_1 + p_2, \quad (27)$$

as can be directly seen by (6)-(7), where the new functions  $\exp(\hat{M})$  and  $\exp(\pm \hat{V})$ , are bounded at  $\sigma = 0$ . It is straightforward to see from (27) that  $\alpha \pm \beta + 1 \geq 0$ . Thus, the singularity term in equation (24) is only given by the factor  $\sigma^{-2}$ .

Observe also that near the singularity  $\sigma = a(u) + b(v) = 0$ , the potential term (25) blows up due to the factor  $\sigma^{-2}$  but it is perfectly smooth on coordinate  $\rho = b(v) - a(u)$ . Thus, the relevant features are related to coordinate  $\sigma$  only. Therefore, we may not expect any physically remarkable difference if we substitute coordinate  $\rho$  in (25) by its constant value  $\rho \equiv \rho_s = b(\hat{s}) - a(\hat{s})$  when the singularity is approached from inside region  $\mathcal{S}$ , i.e. when  $u, v \rightarrow \hat{s}$  such that  $\hat{s}^{n_1} + \hat{s}^{n_2} = 1$ . Then, we will expect a solution of equation (24) of the type  $f(\sigma, \rho) = g(\sigma) \exp(ik_3 \rho)$ , being  $k_3$  a separation constant and where the new function  $g$  depends only on coordinate  $\sigma$  and satisfies the differential equation,

$$g_{,\sigma\sigma} + \left[ \Omega(\sigma, \rho_s) + k_3^2 \right] g = 0.$$

We could now attempt to solve this Schrödinger-type of differential equation. For instance, we could try to find a solution by a WKB expansion. Unfortunately, the initial conditions for equation (24) are given on the boundaries  $\Sigma_I$  and there the coordinates  $\sigma, \rho$  are not well defined. This means that, even though coordinates  $\sigma$  and  $\rho$  would allow us to find a solution near the singularity  $\sigma = 0$ , they cannot be used to find a solution near the boundaries with the plane waves, and therefore they are not convenient for the mode propagation problem. Nevertheless, it will be useful in what follows to know that this type of solution can be found near the singularity.

In fact, we should return to equation (12) in coordinates  $u, v$  and we should recall that we are only interested in finding the field solution in the neighborhood of the collision's center. Thus, the potential term of equation (12) is restricted into region  $\mathcal{S}$ , and the boundary conditions are given on the segments  $\hat{\Sigma}_I$  (see Fig. 2). Then if, in analogy with the change (23), we use the notation  $\sigma = e^{-U}$ , such potential term is simply given by,

$$\Omega(u, v) = \frac{\dot{a}\dot{b}}{4\sigma^2} + \frac{1}{2} e^{-\hat{M}} \sigma^{\alpha-\beta-1} \left( e^{-\hat{V}} k_x^2 + \sigma^{2\beta} e^{\hat{V}} k_y^2 \right), \quad (28)$$

with  $\alpha$ ,  $\beta$  and  $e^{-\hat{V}}$ ,  $e^{-\hat{M}}$  defined in (27). Observe that functions  $e^{-\hat{V}}$ ,  $e^{-\hat{M}}$  and  $\dot{a}(u)\dot{b}(v)$  are perfectly smooth in the entire region  $\mathcal{S}$ . However, (28) blows up when the term  $\sigma = 1 - u^{n_1} - v^{n_2}$  goes to zero at the point  $u = v = \hat{s}$ , where  $\hat{s}^{n_1} + \hat{s}^{n_2} = 1$ . Taking all of this into account, it would be useful to find an adequate approximation for the potential (28) such that: i) it preserves the essential features of (28), ii) it allows us to find a solution of equation (12) which smoothly matches with the boundary conditions at  $\hat{\Sigma}_I$ .

In order to find such an approximation it would be rather convenient to introduce a new set of coordinates naturally adapted to this problem. Recall that coordinates (23) are convenient to solve equation (12) close to the singularity, but they are not well defined globally and particularly on the boundaries with the single plane waves. Nevertheless, since  $\sigma = a(u) + b(v)$  behaves near the singularity in region  $\mathcal{S}$  as,  $\sigma \simeq \dot{a}_s(u - \hat{s}) + \dot{b}_s(v - \hat{s})$ , with  $\dot{a}_s = \dot{a}(\hat{s})$ ,  $\dot{b}_s = \dot{b}(\hat{s})$ , we may define in analogy with (23) a new time-like and space-like coordinates as,

$$t = [\dot{b}_s(v - \hat{s}) + \dot{a}_s(u - \hat{s})] (\dot{a}_s \dot{b}_s)^{-1/2}, \quad z = [\dot{b}_s(v - \hat{s}) - \dot{a}_s(u - \hat{s})] (\dot{a}_s \dot{b}_s)^{-1/2}, \quad (29)$$

where in fact,  $t$  and  $z$  apply globally and close to the singularity in region  $\mathcal{S}$  they behave like coordinates (23). Indeed,  $t \rightarrow \sigma(\dot{a}_s \dot{b}_s)^{-1/2}$  and  $z \rightarrow (\rho - \rho_s)(\dot{a}_s \dot{b}_s)^{-1/2}$  as the singularity  $u = v = \hat{s}$  is approached. The factor  $(\dot{a}_s \dot{b}_s)^{-1/2}$  is used to ensure that such a coordinate change represents a Lorentz transformation.

We can easily find an expression for the function  $\sigma(u, v)$  in coordinates  $t, z$  in the neighborhood of the singularity  $u = v = \hat{s}$  by expanding  $\sigma(u, v)$  in terms of a Taylor series centered in  $u = v = \hat{s}$ , with  $u - \hat{s}$  and  $v - \hat{s}$  expressed, from (29), in terms of coordinates  $t$  and  $z$ . The result is,

$$\sigma(t, z) = t (\dot{a}_s \dot{b}_s)^{1/2} + \sum_{n=2} \left[ \frac{a_s^{(n)}}{\dot{a}_s^n} \left( \frac{t-z}{2} \right)^n + \frac{b_s^{(n)}}{\dot{b}_s^n} \left( \frac{t+z}{2} \right)^n \right] \frac{(\dot{a}_s \dot{b}_s)^{n/2}}{n!}. \quad (30)$$

Observe, however, that in region  $\mathcal{S}$  we expect the physical remarkable effects to occur near the singularity  $u = v = \hat{s}$ , where the potential term (28) grows unbounded as  $\sigma(t, z) \rightarrow 0$ . Recall that in the neighborhood of  $\sigma(t, z) = 0$  the relevant term in (30) is the first order term  $t (\dot{a}_s \dot{b}_s)^{1/2}$ , which is insensitive to coordinate  $z$ . Therefore, we may not expect any physically remarkable difference if instead of working with the function  $\sigma(t, z)$ , which depends on both coordinates  $t$  and  $z$ , we change to a new function  $\sigma(t)$  depending only on coordinate  $t$  and directly derived from (30) by taking  $z = 0$  throughout the entire region  $\mathcal{S}$ , i.e.,

$$\sigma(t) = t (\dot{a}_s \dot{b}_s)^{1/2} + \sum_{n=2} \left( \frac{a_s^{(n)}}{\dot{a}_s^n} + \frac{b_s^{(n)}}{\dot{b}_s^n} \right) \left( \frac{t}{2} \right)^n \frac{(\dot{a}_s \dot{b}_s)^{n/2}}{n!}. \quad (31)$$

Observe also that the rest of the functions which appear in (28), i.e.  $e^{\pm \hat{V}(u,v)}$ ,  $e^{-\hat{M}(u,v)}$  and  $\dot{a}(u)\dot{b}(v)$ , are perfectly smooth all over region  $\mathcal{S}$ . Thus, we may not also expect any physically remarkable difference if we substitute these functions, which also depend on both coordinates  $t$  and  $z$ , by approximate functions depending on coordinate  $t$  alone. The procedure would be equivalent to the one used for  $\sigma(u, v)$ . Namely, given a smooth function  $\psi(u, v)$  in region  $\mathcal{S}$ , we expand it by a Taylor series centered in the singularity  $u = v = \hat{s}$ . Then, we substitute  $u - \hat{s}$ ,  $v - \hat{s}$  by coordinates  $t, z$  using the definition (29) and finally we take  $z = 0$ . The result would simply be,

$$\psi(t) = \psi(\hat{s}, \hat{s}) + \sum_{n=1} \left( \frac{1}{\dot{a}_s} \frac{\partial}{\partial u} + \frac{1}{\dot{b}_s} \frac{\partial}{\partial v} \right)^n \psi(u, v) \Big|_{u=v=\hat{s}} \left( \frac{t}{2} \right)^n \frac{(\dot{a}_s \dot{b}_s)^{n/2}}{n!}. \quad (32)$$

Recall that the boundary conditions for equation (12) restricted in region  $\mathcal{S}$ , lie on the segments  $\hat{\Sigma}_I$  and since the endpoints of these segments are not close to the singularities  $\mathcal{P}$  or  $\mathcal{P}'$ , we may suppose that the boundary conditions are not significantly different from the flat boundary conditions. This fact, together with the prior replacements of  $u, v$ -dependent functions by related  $t$ -dependent functions in the potential term (28), suggest that we may reproduce the main physical features if we replace the four-region mode propagation problem by a much simpler problem. Such a problem would consist in solving equation (12) with a potential term (28) depending on coordinate  $t$  alone and where the boundary conditions would be imposed by the flat Minkowski modes (16) below the hypersurface  $\{T = 0, -\hat{s} < Z < \hat{s}\}$ , where we denote  $T = u + v$ ,  $Z = v - u$  (see Fig. 3). Recall, however, that all this discussion is absolutely non applicable when a solution for equation (12) in the neighborhood of the folding singularities  $\mathcal{P}$  or  $\mathcal{P}'$  is required. This is non applicable because the boundary conditions (21) are unbounded as the folding singularities are approached and therefore these boundary conditions strongly differ from their counterparts (16) in the flat region.

In order to solve this Schrödinger-type problem, rather than relying on the discussed approximations for the exact field equation (12), we will rewrite a new field equation using an adequate approximation for the line element throughout the entire causal past of the collision center. This new approach, which may seem redundant in the case of the *particle production problem*, is absolutely necessary when the renormalization of the stress-energy tensor is discussed. This is essentially because the process of renormalization involves the subtraction of the infinite divergences that arise from the formal definition of the stress-energy tensor, and these divergences can be expressed as entirely geometric terms, which are independent of any possible approximations in the field equation. This means that in order to recover the geometric divergences in the stress-energy tensor, any approximation in the field equation must be related to a suitable approximation in the space-time geometry.

The approximations just discussed above can be essentially recovered by changing the exact colliding plane wave line element throughout the causal region of the collision's center (10) into a related line element. This new line element would basically consist in replacing the metric coefficients  $\sigma(u, v) = e^{-U(u, v)}$ ,  $\sigma(u, v)^{\pm\beta} e^{\pm\hat{V}(u, v)}$ ,  $\sigma(u, v)^{-\alpha} e^{-\hat{M}(u, v)}$  in (10), which depend on both coordinates  $u$  and  $v$ , by new functions of coordinate  $t$  alone in the sense described in (31)-(32), i.e.,

$$d\hat{s}_I^2 = \frac{1}{2} \sigma(t)^\alpha e^{-\hat{M}(t)} (dt^2 - dz^2) - \sigma(t)^{1+\beta} \left( e^{\hat{V}(t)} dx^2 + \sigma(t)^{-2\beta} e^{-\hat{V}(t)} dy^2 \right). \quad (33)$$

We will suppose that the line element (33) applies all over the causal past of the collision center, not only in the interaction region but also through the plane wave regions II and III in the sense of Fig. 3. The plane wave collision starts at  $t = t_0 = -\hat{s} (\dot{b}_s + \dot{a}_s) (\dot{a}_s \dot{b}_s)^{-1/2}$  but to avoid smoothness problems derived from such an approximation, we will suppose that (33) applies exactly on a range  $t_0 + \epsilon < t < 0$ , for a certain  $\epsilon > 0$ . In the range  $t_0 \leq t \leq t_0 + \epsilon$ , as described below, we will interpolate a line element which smoothly matches with the flat space at  $t = t_0$ . Nevertheless, the particular details of this matching will not affect the main physical features.

The exact field equation for this approximate space-time is,

$$(\square + \xi R) \phi = 0, \quad (34)$$

where it is necessary to consider a coupling curvature term in the field equation because, although the exact space-time is a vacuum solution, we have a bounded nonzero value for  $R$  in the approximated space-time. In order to solve this new field equation, we start rewriting the line element (33) in the following general way,

$$ds^2 = (f_1 f_2 f_3) dt^{*2} - \left( \frac{f_1 f_2}{f_3} \right) dz^2 - \left( \frac{f_2 f_3}{f_1} \right) dx^2 - \left( \frac{f_1 f_3}{f_2} \right) dy^2, \quad (35)$$

where the  $f_i$  are functions of coordinate  $t$  alone, which for values of  $t_0 + \epsilon < t < 0$ , can be straightforwardly determined by direct comparison with (33) as

$$f_1^2(t) = \frac{1}{2} e^{-\hat{V}(t) - \hat{M}(t)} \sigma(t)^{(\beta-1)^2/2}, \quad f_2^2(t) = \frac{1}{2} e^{\hat{V}(t) - \hat{M}(t)} \sigma(t)^{(\beta+1)^2/2}, \quad f_3^2(t) = \sigma^2(t), \quad (36)$$

where definitions (27) have been used. For values  $t \leq t_0$  we take  $f_1(t) = f_2(t) = \sqrt{L_1 L_2}$ ,  $f_3(t) = 1$ , which correspond to their values in flat space. Finally, in the interval  $t_0 \leq t \leq t_0 + \epsilon$ , we smoothly interpolate each  $f_i(t)$  ( $i = 1, 2, 3$ ) between these values. Also, in order to prevent singularities in the field equation, we conveniently reparametrize coordinate  $t$ , by  $t^*(t)$ , as follows,

$$\frac{dt^*}{dt} = \frac{1}{f_3(t)}. \quad (37)$$

Now, we use the following ansatz for the field solutions,

$$\phi_k = h(t^*) e^{ik_x x + ik_y y + ik_z z}, \quad (38)$$

where the plane wave factor in coordinates  $x, y$  is related to the translational symmetry of the space-time along the transversal directions  $x, y$ , and the plane wave factor in coordinate  $z$  is just a consequence of our approximation. Then equation (34) directly leads to the following Schrödinger-like differential equation for the function  $h(t^*)$ ,

$$h_{,t^*t^*} + \omega^2(t) h = 0, \quad V(t) \equiv \omega^2(t) = f_0^2(t) + f_1^2(t) k_x^2 + f_2^2(t) k_y^2 + f_3^2(t) k_z^2, \quad (39)$$

where the function  $f_0(t)$  stands for,

$$f_0^2(t) = [f_1(t) f_2(t) f_3(t)] \xi R. \quad (40)$$

Observe from (8) and (27), that the range of the parameter  $\beta$  in (36) is  $\beta^2 < 8$ . Then, we always have that either  $(\beta + 1)^2/4 < 1$  or  $(\beta - 1)^2/4 < 1$ .

Now, it would be rather convenient to discuss whether or not it makes sense to solve the differential equation (39) in terms of a WKB expansion. Observe that, when the parameter  $\beta = \pm 1$  in (36), then the frequency  $\omega(t)$  goes to a non-zero constant value towards the singularity  $\sigma = 0$ . Then the short-wavelength condition, i.e.  $\omega^{-1} d/dt^* \ln \omega \ll 1$ , holds and is particularly accurate near the singularity. Hence, equation (39) admits everywhere a WKB solution.

However, when  $\beta \neq \pm 1$ , the exponents of the  $\sigma$ -terms in the functions  $f_i$  in (36) are all non-zero. Thus, all the functions  $f_i$  vanish at the singularity  $\sigma = 0$ . Also the function  $f_0^2 = \xi R(f_1 f_2 f_3)$  vanishes at  $\sigma = 0$ . This can be easily seen because  $f_0$  can be expressed in terms of  $f_1$ ,  $f_2$  and  $f_3$ , as

$$f_0^2 = \xi \left[ \frac{\ddot{f}_1}{f_1} + \frac{\ddot{f}_2}{f_2} + \frac{\ddot{f}_3}{f_3} - \frac{1}{2} \left( \frac{\dot{f}_1}{f_1} + \frac{\dot{f}_2}{f_2} + \frac{\dot{f}_3}{f_3} \right)^2 \right],$$

and substituting the values (36) we have that  $f_0^2$  is proportional to  $\sigma$ . Therefore, the frequency term  $\omega(t)$  in (39) goes to zero towards the singularity  $\sigma = 0$ . Hence, the short wavelength condition does not hold in the region close to the singularity and thus a WKB solution for equation (39) is not directly appropriate in this region. Nevertheless, we can still use a WKB solution, which at least would be appropriate if we are not close enough to the singularity  $\sigma = 0$  (recall that  $\sigma = 0$  means from (37) that  $t^* \rightarrow \infty$ ). Observe, however, that such a WKB expansion may still give information near the singularity. The reason lies upon the fact that we can interpret this WKB solution as a limit case of a set of WKB expansions which are truly appropriate near the singularity. Such a limit process may be briefly introduced as follows: we add a small quantity to  $V(t)$  in (39). Then, the short wavelength holds and it reduces to  $(dt/dt^*) dV/dt \ll 2\omega^3$ , where now the frequency  $\omega(t)$  is non-zero and  $dt/dt^* = f_3(t) \rightarrow 0$  towards the singularity. Hence, the differential equation (39) admits a WKB solution everywhere. We could then perform the involved calculations and finally set the small quantity that we have introduced in  $V(t)$  to zero. Mathematically, adding this small quantity to  $V(t)$  would be equivalent to subtract a small amount  $\delta > 0$  into the exponents of the  $\sigma$ -terms of functions  $f_1$  and  $f_2$  in (36), i.e.,

$$f_1^2(t) = \frac{1}{2} e^{-\hat{V}(t) - \hat{M}(t)} \sigma(t)^{(\beta-1)^2/2-2\delta}, \quad f_2^2(t) = \frac{1}{2} e^{\hat{V}(t) - \hat{M}(t)} \sigma(t)^{(\beta+1)^2/2-2\delta}, \quad (41)$$

which is also equivalent to slightly change the line element (35) by including a factor  $\sigma(t)^{-2\delta}$  to the metric coefficients  $g_{t^*t^*}$ ,  $g_{zz}$  only. Then, the function  $f_0$  is finite at the singularity and is given by  $f_0^2 = 2\dot{a}_s \dot{b}_s \delta + O(\sigma)$ .

Finally, using a WKB expansion, the mode solutions  $\phi_k$  which reduce to the flat mode solutions in the region prior to the arrival of the waves, are

$$\phi_k = \frac{\hat{\omega}^{1/2}}{\sqrt{(2\pi)^3 2k_- W(t)}} e^{ik_x x + ik_y y + ik_z z - i \int^{t^*} W(\zeta) d\zeta^*}, \quad (42)$$

where we denote  $\hat{\omega}^2 = k_1^2 + k_2^2 + k_3^2$  with  $k_1$  and  $k_2$  given in (26),  $k_3 = k_z$  and where  $W(t)$  stands for an adiabatic series in powers of the time-dependent frequency  $\omega(t)$  of the modes and its derivatives. Up to adiabatic order four (i.e. up to terms involving four derivatives of  $\omega(t)$ )  $W(t)$  it is given by,

$$W(t) = \omega + \frac{A_2}{\omega^3} + \frac{B_2}{\omega^5} + \frac{A_4}{\omega^5} + \frac{B_4}{\omega^7} + \frac{C_4}{\omega^9} + \frac{D_4}{\omega^{11}}, \quad (43)$$

where, using the notation  $\dot{V} \equiv dV/dt^*$ ,

$$A_2 = -\frac{\ddot{V}}{8}, \quad B_2 = \frac{5}{32} \dot{V}^2, \quad (44)$$



$$A_4 = \frac{\ddot{V}}{32}, \quad B_4 = -\frac{28 \dot{V} \ddot{V} + 19 \ddot{V}^2}{128}, \quad C_4 = \frac{221}{258} \dot{V}^2 \ddot{V}, \quad D_4 = -\frac{1105}{2048} \dot{V}^4,$$

and  $A_n, B_n, \dots$  denote the  $n$  adiabatic terms in  $W(t)$ . Up to adiabatic order zero it is simply  $W(t) = \omega(t)$ . Observe the two following facts:

(i) Near the singularity  $\sigma = 0$  we have  $W(t) \simeq \omega(t)$ . This is because the higher adiabatic corrections vanish at the singularity. In fact, for the case  $\beta = \pm 1$  in (36), these higher adiabatic corrections naturally vanish. Also, for the case  $\beta \neq \pm 1$  and under the limit prescription introduced in (41), they also vanish.

(ii) In the flat region prior to the arrival of the waves we have  $W(t) = \hat{\omega} = (k_1^2 + k_2^2 + k_3^2)^{1/2}$ . In that case, since  $f_3 = 1$ , we can use (37) to set  $t^* = t$ , where without loss of generality we choose the value  $t^* = t_0$  at  $t = t_0$ . Therefore, the mode solutions (42) in the flat region reduce to,

$$\phi_k^{\text{IV}} = \frac{1}{\sqrt{(2\pi)^3 2k_-}} e^{ik_x x + ik_y y + ik_z z - i\hat{\omega} t}. \quad (45)$$

To see how (45) are related to the flat modes (16), recall first that the new separation constant  $k_z$  is related to the original  $k_\pm$  by the ordinary null momentum relations, i.e.,

$$\hat{\omega} = \hat{k}_+ + \hat{k}_-, \quad k_z = \hat{k}_+ - \hat{k}_-. \quad (46)$$

In fact, flat modes (16) have the same formal expression as (45) where instead of  $t$  there is a time-like coordinate  $T = v + u$  and instead of  $z$  there is a space-like coordinate  $Z = v - u$ , and coordinates  $(t, z)$  and  $(T, Z)$  are simply related by a Lorentz boost such that,  $t = \gamma(T + \beta Z)$ ,  $z = \gamma(Z + \beta T)$ , with

$$\gamma = \frac{\dot{b}_s + \dot{a}_s}{2\sqrt{\dot{a}_s \dot{b}_s}}, \quad \beta = \frac{\dot{b}_s - \dot{a}_s}{\dot{b}_s + \dot{a}_s}. \quad (47)$$

Unfortunately, we use the same notation  $\beta$  in (47) and in (27), but note that there is not possible confusion.

It is important to understand, however, that we are constructing a set of mode solutions as an adiabatic series in terms of derivatives of the frequency  $\omega(t)$  in the differential equation (39). This procedure is similar but not equivalent to the construction of an *adiabatic vacuum state* where the field modes are expanded as an adiabatic series in terms of the derivatives of the metric coefficients (see for example [48] for details). In fact, observe for instance that the term  $f_0^2(t)$  in (39) involves two derivatives of the metric since it is directly related to the curvature scalar. Thus, it would be a second order term for an eventual adiabatic vacuum, but it is simply a zeroth order term in our adiabatic series in derivatives of  $\omega(t)$ .

## 4 Hadamard function in the interaction region

The key ingredient to calculate the vacuum expectation value of the stress-energy tensor is the *Hadamard function*  $G^{(1)}(x, x')$ , which is defined as the vacuum expectation value of the anticommutator of the field, i.e.,

$$G^{(1)}(x, x') = \langle \{\phi(x), \phi(x')\} \rangle = \sum_k \{u_k(x) u_k^*(x') + u_k(x') u_k^*(x)\}. \quad (48)$$

This Hadamard function contains non-physical divergence terms which can be subtracted by the following point splitting prescription,

$$G_B^{(1)}(x, x') = G^{(1)}(x, x') - S(x, x'), \quad (49)$$

where  $S(x, x')$  is the *midpoint expansion* of a locally constructed quantity commonly referred as a *Hadamard elementary solution* (see for example [49]) and given by

$$\begin{aligned} S(x, x') = & \frac{1}{8\pi^2} \left\{ -\frac{2}{\sigma} - 2\Delta^{(2)}_{\bar{\mu}\bar{\nu}} \frac{\sigma^{\bar{\mu}}\sigma^{\bar{\nu}}}{\sigma} - 2\Delta^{(4)}_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\tau}} \frac{\sigma^{\bar{\mu}}\sigma^{\bar{\nu}}\sigma^{\bar{\rho}}\sigma^{\bar{\tau}}}{\sigma} - a_1^{(0)} \ln(\mu^{-2}\sigma) \right. \\ & \left. - \left[ (a_1^{(0)} \Delta^{(2)}_{\bar{\mu}\bar{\nu}} + a_1^{(2)}_{\bar{\mu}\bar{\nu}}) \sigma^{\bar{\mu}}\sigma^{\bar{\nu}} - \frac{1}{2}a_2^{(0)}\sigma \right] \ln(\mu^{-2}\sigma) - \frac{3}{4}a_2^{(0)}\sigma \right\}, \end{aligned} \quad (50)$$

where the coefficients  $\Delta^{(2)}_{\bar{\mu}\bar{\nu}}$ ,  $\Delta^{(4)}_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\tau}}$ ,  $a_1^{(0)} \dots$  are written in Appendix A. We use the standard definition for the *geodetic biscalar*  $\sigma(x, x') = (1/2)s^2(x, x')$ , being  $s(x, x')$  the proper distance between the points  $x$  and  $x'$  on a non-null geodesic connecting them. Also,  $\sigma_{\bar{\mu}}(x, x') = (\partial/\partial x^{\bar{\mu}})\sigma(x, x')$  is a geodesic tangent vector at the point  $\bar{x}$  with modulus  $s(x, x')$ , being  $\bar{x}$  the *midpoint* between  $x$  and  $x'$  on the geodesic. The parameter  $\mu$  in the logarithmic term of (50) is an arbitrary length parameter, which is related to the two-parameter ambiguity of the point-splitting regularization scheme [49]. Then we can compute  $\langle T_{\mu\nu} \rangle$  by means of the following differential operation,

$$\langle T_{\mu\nu}(x) \rangle = \lim_{x \rightarrow x'} \mathcal{D}_{\mu\nu} G^{(1)}(x, x'), \quad (51)$$

where  $\mathcal{D}_{\mu\nu}$  is a nonlocal differential operator, which in the conformal coupling case ( $\xi = 1/6$ ) is given by,

$$\begin{aligned} \mathcal{D}_{\mu\nu} = & \frac{1}{6} (\nabla_{\mu'} \nabla_{\nu} + \nabla_{\nu'} \nabla_{\mu}) - \frac{1}{24} g_{\mu\nu} (\nabla_{\alpha'} \nabla^{\alpha} + \nabla_{\alpha} \nabla^{\alpha'}) - \\ & \frac{1}{12} (\nabla_{\mu} \nabla_{\nu} + \nabla_{\mu'} \nabla_{\nu'}) + \frac{1}{48} g_{\mu\nu} (\nabla_{\alpha} \nabla^{\alpha} + \nabla_{\alpha'} \nabla^{\alpha'}) - \\ & \frac{1}{12} \left( R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} \right). \end{aligned} \quad (52)$$

However, the above differential operation and its limit have no immediate covariant meaning because  $G^{(1)}(x, x')$  is not an ordinary function but a *biscalar* and the differential operator  $\mathcal{D}_{\mu\nu}$  is *nonlocal*; thus we need to deal with the nonlocal formalism of *bitensors* (see, for example [50, 51] or the Appendix B of reference [33] for a review on this subject).

The regularization procedure (49), however, fails to give a covariantly conserved stress-energy tensor essentially because the locally constructed Hadamard function (50) is not in general symmetric on the endpoints  $x$  and  $x'$  (i.e. it satisfies the field equation at the point  $x$  but fails to satisfy it at  $x'$ ) (see [52] for details). Thus, to ensure covariant conservation, we must introduce an additional prescription:

$$\langle T_{\mu\nu}(x) \rangle = \langle T_{\mu\nu}^B(x) \rangle - \frac{a_2^{(0)}(x)}{64\pi^2} g_{\mu\nu}. \quad (53)$$

Note that this last term is responsible for the trace anomaly in the conformal coupling case, because even though  $\langle T_{\mu\nu}^B(x) \rangle$  has null trace when  $\xi = 1/6$ , the trace of  $\langle T_{\mu\nu}(x) \rangle$  is given by  $\langle T_{\mu}^{\mu} \rangle = -a_2^{(0)}(x)/(16\pi^2)$ . The regularization prescription just given in (53) satisfies the well known four Wald's axioms [49, 53, 54, 55], a set of properties that any physically reasonable expectation value of the stress-energy tensor of a quantum field should satisfy. There is still an ambiguity in this prescription since two independent conserved local curvature terms, which are quadratic in the curvature, can be added to this stress-energy tensor. In particular, the  $\mu$ -parameter ambiguity in (50) is a consequence of this (see [49] for details). Such a two-parameter ambiguity, however, cannot be resolved within the limits of the semiclassical theory, it may be resolved in a complete quantum theory of gravity [49]. Note, however, that in some sense this ambiguity does not affect the knowledge of the matter distribution because a tensor of this kind belongs properly to the left hand side of Einstein equations, i.e. to the geometry rather than to the matter distribution.

## 5 Procedure to compute $\langle T_{\mu\nu} \rangle$

After this preliminary introduction on the point-splitting regularization technique, we may proceed to calculate the Hadamard function  $G^{(1)}(x, x')$  in the interaction region for the initial vacuum state defined by the modes  $\phi_k$ , (42). The Hadamard function can be written as,

$$G^{(1)}(x, x') = \sum_k \phi_k(x) \phi_k^*(x') + c.c. \quad (54)$$

Note that solutions  $\phi_k$  contain the function  $h(t^*)$ , which cannot be calculated analytically but may be approximated up to any adiabatic order as described in (43)-(44). Thus, we have the inherent ambiguity of where to cut the adiabatic series. In fact, this is an asymptotic expansion, which has a well established ultraviolet limit but it may have convergence problems in the low-energy limit. However, observe from (37) and (39) that since: (i)  $dt/dt^* \rightarrow 0$ , (ii)  $V(t) = \omega^2(t)$  is bounded and, under the prescription introduced in (41), is also non-zero towards the singularity, then the adiabatic series (43) reduces to  $W \simeq \omega$  near the singularity. This means that we could cut the adiabatic series (43) at order zero if we were interested in a calculation near the singularity. However, this is only partially true. In fact, it would be true if we were only interested in the *particle production problem*, which essentially would involve the evaluation of a Bogoliubov transformation between two different sets of field modes. Unfortunately it is not sufficient for the calculation of the vacuum expectation value of the stress-energy tensor. This is because  $G^{(1)}$  calculated with  $h(t^*)$  at order zero does not reproduce the short-distance singular structure of a Hadamard elementary solution (50) in the coincidence limit  $x \rightarrow x'$ . The smallest adiabatic order for the function  $h(t^*)$  which we need to recover the singular structure of  $G^{(1)}$  is order four, basically because our adiabatic construction of the mode solutions is similar (but not equivalent) to an *adiabatic vacuum state* (see [48] for details).

In the mode sum (54) we use the shortened notation  $\sum_k \equiv \int_0^\infty dk_-/k_- \int_{-\infty}^\infty dk_x \int_{-\infty}^\infty dk_y$  or equivalently  $\sum_k \equiv (L_1 L_2)^{-1} \int_{-\infty}^\infty dk_1 \int_{-\infty}^\infty dk_2 \int_{-\infty}^\infty dk_3/\hat{\omega}$ , where the change of variables

(46) and the usual notation (26) have been used. Therefore we have,

$$G^{(1)}(x, x') = \frac{1}{2(2\pi)^3 L_1 L_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk_1 dk_2 dk_3}{\sqrt{W(t)W(t')}} \times \\ e^{-i \int_{t^*}^{t'} W(\zeta) d\zeta^* + ik_x(x-x') + ik_y(y-y') + ik_z(z-z')} + c.c. \quad (55)$$

We assume that the points  $x$  and  $x'$  are connected by a non-null geodesic in such a way that they are at the same proper distance  $\epsilon$  from a third midpoint  $\bar{x}$ . We parametrize the geodesic by its proper distance  $\tau$  and with abuse of notation we denote the end points by  $x$  and  $x'$ , which should not be confused with the third component of  $(t, z, x, y)$ . Then we expand the integrand function in powers of  $\epsilon$  and we finally integrate term by term to get an expression up to  $\epsilon^2$ . The details of such a tedious calculation can be found in ref. [34]. The result is,

$$G^{(1)}(x, x') = \bar{A} + \sigma \bar{b} + C_{\bar{\alpha}\bar{\beta}} \sigma^{\bar{\alpha}} \sigma^{\bar{\beta}} + D_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} \sigma^{\bar{\alpha}} \sigma^{\bar{\beta}} \sigma^{\bar{\gamma}} \sigma^{\bar{\delta}} + \frac{1}{8\pi^2} \left\{ -\frac{2}{\sigma} - 2\Delta^{(2)}_{\bar{\mu}\bar{\nu}} \frac{\sigma^{\bar{\mu}} \sigma^{\bar{\nu}}}{\sigma} \right. \\ \left. - 2\Delta^{(4)}_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\tau}} \frac{\sigma^{\bar{\mu}} \sigma^{\bar{\nu}} \sigma^{\bar{\rho}} \sigma^{\bar{\tau}}}{\sigma} - a_1^{(0)} L - \left[ \left( a_1^{(0)} \Delta^{(2)}_{\bar{\mu}\bar{\nu}} + a_1^{(2)}_{\bar{\mu}\bar{\nu}} \right) \sigma^{\bar{\mu}} \sigma^{\bar{\nu}} - \frac{1}{2} a_2^{(0)} \sigma \right] L \right\} \quad (56)$$

where  $L$  is a logarithmic term defined as  $L = 2\gamma + \ln(\sigma \xi R/2)$  and  $\gamma$  is the Euler's constant. All the involved coefficients  $\bar{A}$ ,  $\bar{b}$ ,  $C_{\bar{\alpha}\bar{\beta}} \dots$  depend on which particular space-time, belonging to the Szekeres Class of solutions, we are performing the calculations.

According to (49), the Hadamard function can be regularized using the elementary Hadamard solution (50) and finally the regularized expression for  $G^{(1)}(x, x')$  up to order  $\epsilon^2$  is,

$$G_B^{(1)}(x, x') = \bar{A} + \sigma \bar{B} + C_{\bar{\alpha}\bar{\beta}} \sigma^{\bar{\alpha}} \sigma^{\bar{\beta}} + D_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} \sigma^{\bar{\alpha}} \sigma^{\bar{\beta}} \sigma^{\bar{\gamma}} \sigma^{\bar{\delta}} \\ + \frac{1}{8\pi^2} \left\{ -a_1^{(0)} \hat{L} - \left[ \left( a_1^{(0)} \Delta^{(2)}_{\bar{\mu}\bar{\nu}} + a_1^{(2)}_{\bar{\mu}\bar{\nu}} \right) \sigma^{\bar{\mu}} \sigma^{\bar{\nu}} - \frac{1}{2} a_2^{(0)} \sigma \right] \hat{L} \right\}, \quad (57)$$

where  $\hat{L}$  is a bounded logarithmic term given by,  $\hat{L} = 2\gamma + \ln(\mu^2 \xi R/2)$ ,  $\mu$  being the arbitrary length parameter introduced in (50), and where the coefficient  $\bar{B}$  is  $\bar{B} = b + 3a_2^{(0)}/(32\pi^2)$ . From (57) we can directly read off the regularized mean square field in the “in” vacuum state as  $\langle \phi^2 \rangle = \bar{A}/2 - a_1^{(0)} \hat{L}/(16\pi^2)$ . It is important to remark, however, that the term  $D_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}$  in (57) appears only as a consequence of our approximate procedure of calculating the Hadamard function, i.e. using an adiabatic order four expansion for the initial modes in powers of the mode frequency  $\omega(t)$  and its derivatives. Had we used an exact expression for the initial modes (or an adiabatic vacuum state [48]), such a term would not appear.

Now, to calculate the vacuum expectation value of the stress-energy tensor we have to apply the differential operator (52) to (57). As we have already pointed out, this is not straightforward because we work with nonlocal quantities. Note first that the operator (52) acts on bitensors which depend on the end points  $x$  and  $x'$ , but the expression (57) for  $G_B^{(1)}$  depends on the midpoint  $\bar{x}$ . This means that we need to covariantly expand (57) in

terms of the endpoints  $x$  and  $x'$ . Also, the presence of quartic  $\sigma^\mu$  terms in (57) gives, after differentiation, path dependent terms which must be conveniently averaged. The details of such a calculation may be found for instance in [33, 34]. Then, in the orthonormal basis  $\theta_1 = g_{tt}^{1/2} dt$ ,  $\theta_2 = g_{zz}^{1/2} dz$ ,  $\theta_3 = g_{xx}^{1/2} dx$ ,  $\theta_4 = g_{yy}^{1/2} dy$ , using the trace anomaly prescription (53), we may obtain the expectation values  $\langle T_{\mu\nu} \rangle$  in the conformal coupling case and for values  $t_0 + \epsilon < t < 0$  of coordinate  $t$ . For values of  $t \leq t_0$ ,  $\langle T_{\mu\nu} \rangle = 0$ , and to be consistent with the approximation we have used for the space-time geometry, we should require that the value of  $\langle T_{\mu\nu} \rangle$  goes smoothly to zero as  $t \rightarrow t_0$ . In fact, this can be achieved using an adequate matching of the line element (33) with the flat line element through the interval  $t_0 \leq t \leq t_0 + \epsilon$ .

As a consequence of the logarithmic term in the Hadamard function (57), it will appear a similar term in the stress-energy tensor. The argument of this logarithm depends on the curvature scalar and thus it will grow unbounded as the flat region is approached. However, the coefficient that will appear in front of such a logarithm, depends only on locally constructed curvature terms (as can be seen from (57)). Therefore, with an adequate matching of the space-time geometry, this coefficient will also smoothly vanish towards the flat space region, below  $t = t_0$ . The details of such a matching, however, will not affect the main features of the stress-energy tensor, particularly when the singularity is approached.

Recall that the components of  $\langle T_{\mu\nu} \rangle$  are expressed in the orthonormal tetrad basis  $(\theta_1, \theta_2, \theta_3, \theta_4)$  defined upon coordinates  $(t, z, x, y)$ , with  $t$  and  $z$  defined in (29). These coordinates allow us to describe symmetrically the head on plane wave collision. Thus, written in these coordinates, the stress energy tensor is diagonal. However, an observer at rest at the collision center is better described by means of the *comoving coordinates*  $T = v + u$ ,  $Z = v - u$ , such that  $Z = 0$  for the observer's worldline. Observe that the coordinate systems  $(t, z)$  and  $(T, Z)$  are related by the Lorentz transformation (47). Therefore, the stress-energy tensor in the observer's orthonormal comoving frame will be determined by expressing  $\langle T_{\mu\nu} \rangle$  in the orthonormal tetrad basis  $\theta_{\hat{1}} = g_{TT}^{1/2} dT$ ,  $\theta_{\hat{2}} = g_{ZZ}^{1/2} dZ$ ,  $\theta_{\hat{3}} = g_{xx}^{1/2} dx$ ,  $\theta_{\hat{4}} = g_{yy}^{1/2} dy$ , defined upon coordinates  $(T, Z, x, y)$ . This can be directly achieved by transforming  $\langle T_{\mu\nu} \rangle$ , expressed in the basis  $(\theta_1, \theta_2, \theta_3, \theta_4)$ , by the Lorentz transformation (47), i.e.,

$$\begin{aligned} \langle T_{\hat{1}\hat{1}} \rangle &= \gamma^2 (\langle T_{11} \rangle + \beta^2 \langle T_{22} \rangle), & \langle T_{\hat{2}\hat{2}} \rangle &= \gamma^2 (\langle T_{22} \rangle + \beta^2 \langle T_{11} \rangle), \\ \langle T_{\hat{1}\hat{2}} \rangle &= \gamma^2 \beta (\langle T_{11} \rangle + \langle T_{22} \rangle). \end{aligned} \quad (58)$$

Observe that in comoving coordinates  $(T, Z)$  the collision is not symmetrical anymore and the stress-energy tensor acquires a non-diagonal term  $\langle T_{\hat{1}\hat{2}} \rangle$ , which indicates the flux of momentum in the  $Z$  direction.

## 6 Conclusions

We have given a procedure to calculate the expectation value of the stress-energy tensor of a massless scalar field in a family of space-times representing the head on collision of two gravitational plane waves throughout the causal past of the collision center and in the field state which corresponds to the physical vacuum state before the collision. We have considered this particular region essentially because, as we pointed out in a previous work

[34, 35], we could introduce a suitable approximation to the space-time metric (see Fig. 3) which not only allow us to dramatically simplify the calculations but also to keep unchanged the main physical features, in particular the behaviour of the stress-energy tensor near the singularity of the interaction region.

Unfortunately, a general expression for the stress-energy tensor has not been found. The reason is because the approximation (33) that we have used in the line element, throughout the causal past of the collision center, depends on which particular solution, from the Szekeres Class of solutions, we are considering. Nevertheless, one could expect that near the singularity the main physical effects would be related to the exponents of the singular  $\sigma(t)$  terms in the line element (33). Thus, the main contribution to the stress-energy tensor would be essentially related to these exponents and only trivially sensitive to the remaining smooth functions.

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## A Some midpoint expansions

The coefficients for the *midpoint expansion* of the locally constructed Hadamard function (50) are:

$$\begin{aligned}
a_1^{(0)} &= -R \left( \xi - \frac{1}{6} \right), \quad \Delta^{(2)}_{\mu\nu} = \frac{1}{12} R_{\mu\nu}, \\
a_2^{(0)} &= \frac{1}{2} \left( \frac{1}{6} - \xi \right)^2 R^2 + \frac{1}{6} \left( \frac{1}{5} - \xi \right) R_{;\alpha}{}^\alpha - \frac{1}{180} R^{\alpha\beta} R_{\alpha\beta} + \frac{1}{180} R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}, \\
a_1^{(2)}{}_{\mu\nu} &= \frac{1}{24} \left( \frac{1}{10} - \xi \right) R_{;\mu\nu} + \frac{1}{120} R_{\mu\nu;\alpha}{}^\alpha - \frac{1}{90} R^\alpha{}_\mu R_{\alpha\nu} + \\
&\quad \frac{1}{180} R^{\alpha\beta} R_{\alpha\mu\beta\nu} + \frac{1}{180} R^{\alpha\beta\gamma}{}_\mu R_{\alpha\beta\gamma\nu}, \\
\Delta^{(4)}{}_{\mu\nu\rho\tau} &= \frac{3}{160} R_{\mu\nu;\rho\tau} + \frac{1}{288} R_{\mu\nu} R_{\rho\tau} + \frac{1}{360} R^\alpha{}_\mu{}^\beta{}_\nu R_{\alpha\rho\beta\tau}.
\end{aligned}$$

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## Figure captions

**Fig. 1** The colliding plane wave space-time consists of two approaching waves, regions II and III, in a flat background, region IV, and an interaction region, region I. The two waves move in the direction of two null coordinates  $u$  and  $v$ . The four space-time regions are separated by the two null wave fronts  $u = 0$  and  $v = 0$ . The boundary between regions I and II is  $\{0 \leq u < 1, v = 0\}$ , the boundary between regions I and III is  $\{u = 0, 0 \leq v < 1\}$ , and the boundary of regions II and III with region IV is  $\Sigma = \{u \leq 0, v = 0\} \cup \{u = 0, v \leq 0\}$ . Region I meets region IV only at the surface  $u = v = 0$ . The singularity in the region I corresponds to the hypersurface  $u^{n_1} + v^{n_2} = 1$  and plane wave regions II and III meet such a singularity only at  $\mathcal{P} = \{u = 1, v = 0\}$  and  $\mathcal{P}' = \{u = 0, v = 1\}$  respectively. The hypersurfaces  $u = 1$  in region II and  $v = 1$  in region III are a type of topological singularities commonly referred as folding singularities and they must be identified with  $\mathcal{P}$  and  $\mathcal{P}'$  respectively.

**Fig. 2** The subset of Cauchy data which affects the evolution of the quantum field along the center  $u = v$  of the plane wave collision lies on the segments  $\hat{\Sigma}_I = \{0 \leq u < \hat{s}, v = 0\} \cup \{u = 0, 0 \leq v < \hat{s}\}$ , where  $\hat{s}^{n_1} + \hat{s}^{n_2} = 1$ . Region  $\mathcal{S}$  is the causal future of this Cauchy data (or equivalently, the causal past of the collision center).

**Fig. 3** We change the *mode propagation problem* for the plane wave collision, in the causal past of the collision center, region  $\mathcal{S}$ , by a much simpler Schrödinger-type problem which consists in: i) performing a Lorentz transformation in order to work with an adequate set of coordinates  $(t, z)$ , ii) eliminating the dependence of the field equation on coordinate  $z$

by taking  $z = 0$  and iii) substituting the Cauchy data, which come from the single plane wave regions, on segments  $\hat{\Sigma}_I = \{0 \leq u < \hat{s}, v = 0\} \cup \{u = 0, 0 \leq v < \hat{s}\}$  by much simpler Minkowski Cauchy data. This procedure is essentially equivalent to: i) writing the line element in coordinates  $(t, z)$ , ii) modifying the space-time geometry in the causal past of the collision center by eliminating the dependence on coordinate  $z$  in the line element, setting  $z = 0$  and iii) smoothly matching this line element, through plane wave regions II and III, with the flat spacetime below the segment  $\{T = 0, -\hat{s} < Z < \hat{s}\}$ , where we denote  $T = u + v$ ,  $Z = v - u$ .





